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TRANSIENT RESPONSE OF A CYLINDRICAL SHELL OF FINITE LENGTH TO TRANSVERSE IMPACT

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Abstract—An exact analytical solution within the linear framework is developed for the transient dynamic impact response of a two-end simply supported cylindrical shell of finite length. The solution obtained is shown to consist of a remarkably simple analytical expression. Several examples of impact loadings are calculated. The results show excellent agreement with those of the computer code ABAQUS.

NOTATION

$\mathbf{A}$	coefficient matrix of governing equations
a_n and c_m	coefficients of Fourier series
a_{ij} $(i, j = 1, 3)$	elements of matrix $\mathbf{A}$
b_i $(i = 1, 9)$	coefficients of expansion of polynomial
$c = \left[\frac{E}{(1-\nu^2)\rho}\right]^{1/2}$	stress wave velocity in material
D_1, D_2	membrane stiffness and bending stiffness, respectively
$\mathbf{E}$	unit matrix
E, ν	Young's modulus and Poisson's ratio, respectively
h', h	thickness of shell
I	peak impulse intensity per unit area
$\bar{I} = \frac{Ic(1-\nu^2)}{Eh}$	dimensionless peak impulse intensity per unit area
k_1, k_2, k_3, α_1, α_2	dimensionless frequencies
l	length of cylindrical shell
m, n	Fourier series parameters
M_1	axial bending moment component
M_2	circumferential bending moment component
N_1	axial membrane force component
N_2	circumferential membrane force component
p', p	distributed load
R	radius of cylindrical shell
s	Laplace transform parameter
t	time variable
u', u	axial displacement of the middle surface
v', v	circumferential displacement of the middle surface
w', w	radial displacement of the middle surface
x, β, z	axial, circumferential and radial coordinates, respectively
ρ	density of material
$\sigma_1^{\pm}$	axial stresses at inner and outer surfaces of shell
$\sigma_2^{\pm}$	circumferential stresses at inner and outer surfaces of shell
$\bar{\sigma}_1^{\pm}$	dimensionless axial stresses
$\bar{\sigma}_2^{\pm}$	dimensionless circumferential stresses
$\tau = \frac{ct}{R}$	dimensionless time variable
τ_1	dimensionless time duration of impulse
$\lambda_m = \frac{m\pi R}{l}$	Fourier series parameter.

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1. INTRODUCTION

In recent years, concern has been expressed regarding the dynamic response analysis of structures sustaining impact loading. Cylindrical shells, as a type of important engineering structure, have been extensively investigated. In retrospect, some analytical solutions of the transient dynamic response of cylinders were obtained in the 60s and early 70s (Humphreys and Winter, 1965; Pawlik and Reismann, 1973). Since then, due to the rapid development of numerical techniques it seems that little progress has been achieved in the analytical solution of this problem. Although numerical techniques are efficient and can be used to solve various complex dynamic response problems of structures, they need large computing power and the computer running time may be such as to make it unacceptable for small capacity computers. Hence, it is desirable to obtain efficient and simple exact solutions for these kinds of impact response problems.

In problems of transient dynamic response of thin-walled cylindrical shells, three approximate shell theories have been commonly used (Reismann and Pawlik, 1968): (a) membrane theory; (b) combined membrane and bending theory and (c) improved theory (including shear deformation and rotation inertia). The analysis is limited to the solution of partial differential equations with only two independent variables: the angular coordinate β and time t. For the membrane theory, the circumferential stress is assumed to be constant throughout the thickness of the shell and therefore bending moments vanish. Travelling wave solutions for membrane stresses were presented in some published papers (Payton, 1961; Forrestal and Alzheimer, 1969).

The combined membrane and bending shell theory (Flügge, 1962) uses the Kirchhoff hypothesis and allows for variation of the curcumferential stress throughout the thickness of the shell, resulting in bending moments as well as membrane forces. A solution for both membrane and bending stresses based on this theory was presented by Humphreys and Winter (1965). In the improved theory, straight-line elements originally normal to the median surface are allowed to rotate, and a new parameter is introduced to account for the effect of transverse shear deformation. The improved theory, which is the best of the aforementioned three, accounts for membrane forces, bending moments, transverse shear deformation and the effects of rotatory inertia (Herrmann and Mirsky, 1957; Goodier and McIvor, 1962, 1964). It can be reduced to the combined membrane and bending theory if rotatory inertia and transverse shear deformation effects are neglected.

All these two-dimensional thin cylindrical shell theories are approximate. Although they have the advantage of simplicity, they suffer from one major disadvantage: they ignore the surface traction on the two ends of the shell so that they cannot be applied to shells of finite length.

In this paper, an exact analytical solution is presented for the transient dynamic response of a thin-walled, elastic, cylindrical shell of finite length under transverse impact loading. The solution of partial differential equations, based on the Kirchhoff hypothesis, contains three independent variables: the axial coordinate x, the circumferential coordinate β and time T. The general solutions for displacements and membrane and bending stresses are presented in closed form as a double trigonometric series. Two cases of impulsive loading are calculated. In the first example, the loading is assumed to be a rectangular impulse distributed over half the shell circumference along the shell length. In addition, the transient dynamic response of a cylindrical shell subjected to lateral, localized, distributed impact loading is considered in the last example. Comparison of the present results with the results of Humphreys and Winter (1965) and the computer code ABAQUS (Hibbitt *et al.*, 1988) show excellent agreement.

2. OUTLINE OF METHOD OF ANALYSIS

Consider a thin-walled circular cylindrical shell of finite length made of an isotropic, linearly elastic material. It is simply supported along the edges $x = 0$ and $x = l$, and subjected to transverse impulsive loading. The shell is referred to as a right-hand system of orthogonal, curvilinear coordinates x, β and z which represent the axial coordinate, circumferential coordinate and the coordinate perpendicular to the middle surface of the shell (Fig. 1), respectively. To simplify later derivations, we first introduce the following dimensionless variables

$$u = \frac{u'}{R}, \quad v = \frac{v'}{R}, \quad w = \frac{w'}{R}, \quad x = \frac{x'}{R}$$

$$h = \frac{h'}{R}, \quad p = \frac{(1-\nu^2)R}{Eh'}p', \quad \tau = \frac{ct}{R} = \frac{t}{R}\sqrt{\frac{E}{\rho(1-\nu^2)}} \tag{1}$$

where the prime (′) denotes the actual variable. By adding the inertia terms, the dimensionless forms of the governing equations (Timoshenko and Woinosky-Krieger, 1959, pp. 522–523) are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial \beta^2}\right)u + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial \beta} - \nu\frac{\partial w}{\partial x} = \ddot{u} \tag{2}$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial \beta} + \left(\frac{\partial^2}{\partial \beta^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right)v - \frac{\partial w}{\partial \beta} = \ddot{v} \tag{3}$$

$$\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \beta} - w - \frac{h^2}{12}\nabla^2 w = \ddot{w} - p, \tag{4}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \beta^2}\right)^2$$

and the dots denote differentiation with respect to time τ.

We assume that the two ends of the cylindrical shell are simply-supported, so the corresponding boundary conditions and initial conditions can be written as follows:

$$(v, w, N_1, M_1)|_{x=0} = 0, \quad (\dot{u}, \dot{v}, \dot{w})|_{\tau=0} = 0$$
$$(v, w, N_1, M_1)|_{x=l} = 0, \quad (u, v, w)|_{\tau=0} = 0. \tag{5}$$

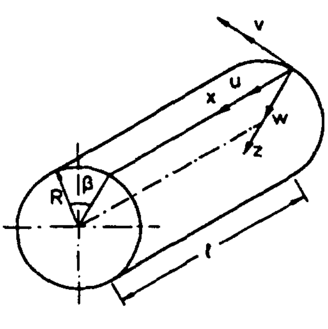

Fig. 1. Geometry of a cylindrical shell associated with Cartesian coordinates.

The boundary conditions suggest the use of the Fourier transform. So, we define

$$\phi_1(m,\beta,\tau) = \frac{2R}{l}\int_0^{l/R} u(x,\beta,\tau)\cos\lambda_m x\,dx$$

$$\phi_2(m,\beta,\tau) = \frac{2R}{l}\int_0^{l/R} v(x,\beta,\tau)\sin\lambda_m x\,dx$$

$$\phi_3(m,\beta,\tau) = \frac{2R}{l}\int_0^{l/R} w(x,\beta,\tau)\sin\lambda_m x\,dx$$

$$\phi_4(m,\beta,\tau) = \frac{2R}{l}\int_0^{l/R} p(x,\beta,\tau)\sin\lambda_m x\,dx \tag{6}$$

where

$$\lambda_m = \frac{m\pi R}{l},$$

then

$$u(x,\beta,\tau) = \sum_{m=0}^{\infty} \phi_1(m,\beta,\tau)\cos\lambda_m x$$

$$v(x,\beta,\tau) = \sum_{m=1}^{\infty} \phi_2(m,\beta,\tau)\sin\lambda_m x$$

$$w(x,\beta,\tau) = \sum_{m=1}^{\infty} \phi_3(m,\beta,\tau)\sin\lambda_m x$$

$$p(x,\beta,\tau) = \sum_{m=1}^{\infty} \phi_4(m,\beta,\tau)\sin\lambda_m x. \tag{7}$$

With further expansion of the coefficients of the above series with respect to β, let

$$U(n,m,\tau) = \frac{2}{\pi}\int_0^{\pi} \phi_1(m,\beta,\tau)\cos n\beta\,d\beta$$

$$V(n,m,\tau) = \frac{2}{\pi}\int_0^{\pi} \phi_2(m,\beta,\tau)\sin n\beta\,d\beta$$

$$W(n,m,\tau) = \frac{2}{\pi}\int_0^{\pi} \phi_3(m,\beta,\tau)\cos n\beta\,d\beta$$

$$P(n,m,\tau) = \frac{2}{\pi}\int_0^{\pi} \phi_4(m,\beta,\tau)\cos n\beta\,d\beta. \tag{8}$$

Finally, the three displacement components u, v, w and distributed load p, can be expressed in double trigonometric series form, respectively

$$u(x,\beta,\tau) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} U(n,m,\tau)\cos\lambda_m x\cos n\beta$$

$$v(x,\beta,\tau) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} V(n,m,\tau)\sin\lambda_m x\sin n\beta$$

$$w(x,\beta,\tau) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} W(n,m,\tau)\sin\lambda_m x\cos n\beta$$

$$p(x,\beta,\tau) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} P(n,m,\tau)\sin\lambda_m x\cos n\beta. \tag{9}$$

One can thus prove that eqn (9) satisfies the boundary conditions (5). Substituting eqn (9) into the dimensionless governing eqn (4), we get

$$\mathbf{AW} = \mathbf{W}_{\tau\tau} - \mathbf{P}. \tag{10}$$

Here, the boldfaced letters denote the matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \mathbf{W} = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \quad \mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$$

where the elements of matrix A are

$$\begin{aligned} &a_{11} = -\lambda_m^2 - \tfrac{1}{2}(1-\nu)n^2; \quad && a_{12} = \tfrac{1}{2}(1+\nu)n\lambda_m; \quad && a_{13} = -\nu\lambda_m \\ &a_{21} = a_{12}; && a_{22} = -n^2 - \tfrac{1}{2}(1-\nu)\lambda_m^2; && a_{23} = n \\ &a_{31} = a_{13}; && a_{32} = a_{23}; && a_{33} = -1 - \frac{h^2}{12}(n^4 + 2\lambda_m^2 n^2 + \lambda_m^4). \end{aligned} \tag{11}$$

The general solution (see the Appendix) of eqn (10) is given by

$$\begin{aligned} U &= \int_0^\tau \left[\frac{b_1}{k_1}\sin k_1(\tau-s) + \frac{b_2}{k_2}\sin k_2(\tau-s) + \frac{b_3}{k_3}\sin k_3(\tau-s)\right] P(s)\,ds \\ V &= \int_0^\tau \left[\frac{b_4}{k_1}\sin k_1(\tau-s) + \frac{b_5}{k_2}\sin k_2(\tau-s) + \frac{b_6}{k_3}\sin k_3(\tau-s)\right] P(s)\,ds \\ W &= \int_0^\tau \left[\frac{b_7}{k_1}\sin k_1(\tau-s) + \frac{b_8}{k_2}\sin k_2(\tau-s) + \frac{b_9}{k_3}\sin k_3(\tau-s)\right] P(s)\,ds, \end{aligned} \tag{12}$$

where b_i $(i = 1, 9)$ and k_j $(j = 1, 2, 3)$ are given in the Appendix. In addition, for axisymmetric conditions when $n = 0$, eqn (10) can be reduced to the following form

$$\begin{pmatrix} \lambda_m^2 + s^2 & \nu\lambda_m \\ \nu\lambda_m & 1 + \frac{h^2}{12}\lambda_m^4 + s^2 \end{pmatrix} \begin{pmatrix} \bar{U}_0 \\ \bar{W}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{P}_0 \end{pmatrix}. \tag{13}$$

By means of the Laplace inverse integral transform, we obtain

$$\begin{aligned} U_0 &= \int_0^\tau \frac{\nu\lambda_m}{Q}\left[\frac{\sin\alpha_2(\tau-s)}{\alpha_2} - \frac{\sin\alpha_1(\tau-s)}{\alpha_1}\right] P(s)\,ds \\ W_0 &= \int_0^\tau \left[\frac{(\lambda_m^2-\alpha_1^2)}{Q\alpha_1}\sin\alpha_1(\tau-s) - \frac{(\lambda_m^2-\alpha_2^2)}{Q\alpha_2}\sin\alpha_2(\tau-s)\right] P(s)\,ds \\ V_0 &= 0, \end{aligned} \tag{14}$$

where

$$\alpha_1^2 = \frac{1}{2}\left(1 + \lambda_m^2 + \frac{1}{12}h^2\lambda_m^4\right) + \frac{1}{2}\left[(\lambda_m^2-1)^2 + 4\nu^2\lambda_m^2 - \frac{1}{6}h^2\lambda_m^4(\lambda_m^2-1) + \frac{1}{144}h^4\lambda_m^8\right]^{1/2}$$

$$\alpha_2^2 = \frac{1}{2}\left(1 + \lambda_m^2 + \frac{1}{12}h^2\lambda_m^4\right) - \frac{1}{2}\left[(\lambda_m^2-1)^2 + 4\nu^2\lambda_m^2 - \frac{1}{6}h^2\lambda_m^4(\lambda_m^2-1) + \frac{1}{144}h^4\lambda_m^8\right]^{1/2}$$

$$Q = \alpha_2^2 - \alpha_1^2 = -\left[(\lambda_m^2-1)^2 + 4\nu^2\lambda_m^2 - \frac{1}{6}h^2\lambda_m^4(\lambda_m^2-1) + \frac{1}{144}h^4\lambda_m^8\right]^{1/2}.$$

Finally, the displacements are obtained in terms of the series defined in eqns (9). Therefore, the generalized forces, i.e. the membrane forces and the bending moments, can easily be obtained from the associated force–displacement relations

$$N_1 = D_1\left[\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial \beta} - w\right)\right]$$

$$N_2 = D_1\left[\left(\frac{\partial v}{\partial \beta} - w\right) + \nu\frac{\partial u}{\partial x}\right]$$

$$M_1 = -D_2\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial \beta^2}\right)$$

$$M_2 = -D_2\left(\frac{\partial^2 w}{\partial \beta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{15}$$

where

$$D_1 = \frac{Eh'}{1-\nu^2}, \quad D_2 = \frac{Eh'^3}{12(1-\nu^2)} \tag{16}$$

are the membrane stiffness and bending stiffness of the shell, respectively. The stresses in the shell at the outer and inner surfaces can be obtained by substituting eqn (15) into

$$\sigma_1|_{z=\pm h/2} = \frac{N_1}{h'} \pm \frac{6M_1}{h'^2}$$

$$\sigma_2|_{z=\pm h/2} = \frac{N_2}{h'} \pm \frac{6M_2}{h'^2}, \tag{17}$$

yielding

$$\frac{1-\nu^2}{E}\sigma_1|_{z=\pm h/2} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{-\lambda_m U + \nu n V - \left[\nu \mp \frac{h}{2}(\lambda^2 + \nu n^2)\right]W\right\}\cos n\beta \sin \lambda_m x$$

$$\frac{1-\nu^2}{E}\sigma_2|_{z=\pm h/2} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{-\nu\lambda_m U + nV - \left[1 \mp \frac{h}{2}(\nu\lambda^2 + n^2)\right]W\right\}\cos n\beta \sin \lambda_m x. \tag{18}$$

3. EXAMPLES

3.1. *Example A*

Consider the case when the shell sustains a rectangular pressure impulse of duration t_1 distributed as shown in Fig. 2. The analytical expression of the loading is

$$p'(x,\beta,t) = \begin{cases} p'_0 \cos\beta, & 0 \leqslant t \leqslant t_1, -\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}, 0 \leqslant x \leqslant l; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

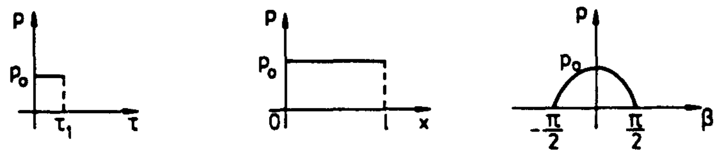

Fig. 2. Impact loading as a function of time and position (Example A).

Defining a dimensionless impulse per unit area

$$\bar{I} = \frac{Ic(1-\nu^2)}{Eh'} = p_0\tau_1$$

where

$$I = \int_0^{t_1} p_0' \, dt = p_0' t_1, \quad \tau_1 = \frac{ct_1}{R}, \quad p_0 = \frac{(1-\nu^2)R}{Eh'} p_0'.$$

and the wave speed

$$c = \sqrt{\frac{E}{\rho(1-\nu^2)}}.$$

Then, the dimensionless transformed form of loading is

$$p(x,\beta,\tau) = \begin{cases} \dfrac{\bar{I}}{\tau_1}\cos\beta, & -\dfrac{\pi}{2} \leqslant \beta \leqslant \dfrac{\pi}{2}, \; 0 \leqslant x \leqslant l, \; 0 \leqslant \tau \leqslant \tau_1; \\ 0, & |\beta| > \dfrac{\pi}{2}, \; \tau > \tau_1, \end{cases} \tag{20}$$

in which

$$\cos\beta = \sum_{n=0}^{L} \sum_{m=1}^{L} a_n c_m \cos n\beta \sin\lambda_m x$$

where the Fourier series coefficients are given by

$$a_0 = \frac{1}{\pi}; \quad a_1 = \frac{1}{2}$$

$$a_n = \begin{cases} -\dfrac{2(-1)^{n/2}}{\pi(n^2-1)}, & n = 2, 4, \ldots \\ 0, & n = 3, 5, \ldots \end{cases}$$

$$c_m = \begin{cases} \dfrac{4}{\pi m}, & m = 1, 3, \ldots \\ 0, & m = 2, 4, \ldots. \end{cases}$$

Now, combining eqn (12) and eqn (14) and integrating, we can obtain the displacement response results. For $\tau \leqslant \tau_1$,

$$U_0 = \frac{\bar{I}a_0 c_m \nu\lambda_m}{\tau_1 Q}\left(\frac{1-\cos\alpha_2\tau}{\alpha_2^2} - \frac{1-\cos\alpha_1\tau}{\alpha_1^2}\right)$$

$$W_0 = \frac{\bar{I}a_0 c_m}{\tau_1 Q}\left[\frac{\lambda_m^2-\alpha_1^2}{\alpha_1^2}(1-\cos\alpha_1\tau) - \frac{\lambda_m^2-\alpha_2^2}{\alpha_2^2}(1-\cos\alpha_2\tau)\right]$$

$$U = \frac{\bar{I}a_n c_m}{\tau_1}\left[\frac{b_1}{k_1^2}(1-\cos k_1\tau) + \frac{b_2}{k_2^2}(1-\cos k_2\tau) + \frac{b_3}{k_3^2}(1-\cos k_3\tau)\right]$$

$$V = \frac{\bar{I}a_n c_m}{\tau_1}\left[\frac{b_4}{k_1^2}(1-\cos k_1\tau) + \frac{b_5}{k_2^2}(1-\cos k_2\tau) + \frac{b_6}{k_3^2}(1-\cos k_3\tau)\right]$$

$$W = \frac{\bar{I}a_n c_m}{\tau_1}\left[\frac{b_7}{k_1^2}(1-\cos k_1\tau) + \frac{b_8}{k_2^2}(1-\cos k_2\tau) + \frac{b_9}{k_3^2}(1-\cos k_3\tau)\right]. \tag{21}$$

For $\tau \geqslant \tau_1$, the response is given by

$$U_0 = \frac{\bar{I}a_0 c_m \nu\lambda_m}{Q}[(S_{21}\cos\alpha_2\tau + S_{22}\sin\alpha_2\tau) - (S_{11}\cos\alpha_1\tau + S_{12}\sin\alpha_1\tau)]$$

$$W_0 = \frac{\bar{I}a_0 c_m}{Q}[(\lambda_m^2-\alpha_1^2)(S_{11}\cos\alpha_1\tau + S_{12}\sin\alpha_1\tau) - (\lambda_m^2-\alpha_2^2)(S_{21}\cos\alpha_2\tau + S_{22}\sin\alpha_2\tau)]$$

$$U = \bar{I}a_n c_m[b_1(J_{11}\cos k_1\tau + J_{12}\sin k_1\tau) + b_2(J_{21}\cos k_2\tau + J_{22}\sin k_2\tau) + b_3(J_{31}\cos k_3\tau + J_{32}\sin k_3\tau)]$$

$$V = \bar{I}a_n c_m[b_4(J_{11}\cos k_1\tau + J_{12}\sin k_1\tau) + b_5(J_{21}\cos k_2\tau + J_{22}\sin k_2\tau) + b_6(J_{31}\cos k_3\tau + J_{32}\sin k_3\tau)]$$

$$W = \bar{I}a_n c_m[b_7(J_{11}\cos k_1\tau + J_{12}\sin k_1\tau) + b_8(J_{21}\cos k_2\tau + J_{22}\sin k_2\tau) + b_9(J_{31}\cos k_3\tau + J_{32}\sin k_3\tau)], \tag{22}$$

where

$$\begin{cases} S_{j1} = \dfrac{\cos\alpha_j\tau_1 - 1}{\alpha_j^2\tau_1} \\ S_{j2} = \dfrac{\sin\alpha_j\tau_1}{\alpha_j^2\tau_1} \end{cases} \quad (j = 1,2) \qquad \begin{cases} J_{i1} = \dfrac{\cos k_i\tau_1 - 1}{k_i^2\tau_1} \\ J_{i2} = \dfrac{\sin k_i\tau_1}{k_i^2\tau_1} \end{cases} \quad (i = 1,2,3).$$

Now we can define

$$f_1^{\pm}(y_1, y_2, y_3, n) = -\lambda_m y_1 + \nu n y_2 - \nu y_3 \pm \frac{h}{2}(\lambda_m^2 + \nu n^2)y_3$$

$$f_2^{\pm}(y_1, y_2, y_3, n) = -\nu\lambda_m y_1 + n y_2 - y_3 \pm \frac{h}{2}(\nu\lambda_m^2 + n^2)y_3 \tag{23}$$

then, the stress responses, after the impact has vanished, i.e. for $\tau > \tau_1$ (the region of most interest for short pulses), are

$$\bar{\sigma}_1^{\pm} \equiv \frac{h}{Ic}\sigma_1|_{z=\pm h/2}$$

$$= \frac{4}{\pi^2}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{mQ}\Big\{-\nu\lambda_m^2[(S_{21}\cos\alpha_2\tau+S_{22}\sin\alpha_2\tau)-(S_{11}\cos\alpha_1\tau+S_{12}\sin\alpha_1\tau)]$$

$$-\left(\nu\mp\frac{h}{2}\lambda_m^2\right)[(\lambda_m^2-\alpha_1^2)(S_{11}\cos\alpha_1\tau+S_{12}\sin\alpha_1\tau)$$

$$-(\lambda_m^2-\alpha_2^2)(S_{21}\cos\alpha_2\tau+S_{22}\sin\alpha_2\tau)]\Big\}\sin\lambda_m x$$

$$+\frac{2}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m}[f_1^{\pm}(b_1,b_4,b_7,1)(J_{11}\cos k_1\tau+J_{12}\sin k_1\tau)$$

$$+f_1^{\pm}(b_2,b_5,b_8,1)+(J_{21}\cos k_2\tau+J_{22}\sin k_2\tau)$$

$$+f_1^{\pm}(b_3,b_6,b_9,1)(J_{31}\cos k_3\tau+J_{32}\sin k_3\tau)]\cos\beta\sin\lambda_m x$$

$$-\frac{8}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{n/2}}{m(n^2-1)}[f_1^{\pm}(b_1,b_4,b_7,n)(J_{11}\cos k_1\tau+J_{12}\sin k_1\tau)$$

$$+f_1^{\pm}(b_2,b_5,b_8,n)(J_{21}\cos k_2\tau+J_{22}\sin k_2\tau)$$

$$+f_1^{\pm}(b_3,b_6,b_9,n)(J_{31}\cos k_3\tau+J_{32}\sin k_3\tau)]\cos n\beta\sin\lambda_m x. \tag{24}$$

$$\bar{\sigma}_2^{\pm} \equiv \frac{h}{Ic}\sigma_2|_{z=\pm h/2}$$

$$= \frac{4}{\pi^2}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{mQ}\Big\{-\nu^2\lambda_m^2[(S_{21}\cos\alpha_2\tau+S_{22}\sin\alpha_2\tau)-(S_{11}\cos\alpha_1\tau+S_{12}\sin\alpha_1\tau)]$$

$$-\left(1\mp\frac{h}{2}\nu\lambda_m^2\right)[(\lambda_m^2-\alpha_1^2)(S_{11}\cos\alpha_1\tau+S_{12}\sin\alpha_1\tau)$$

$$-(\lambda_m^2-\alpha_2^2)(S_{21}\cos\alpha_2\tau+S_{22}\sin\alpha_2\tau)]\Big\}\sin\lambda_m x$$

$$+\frac{2}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m}[f_2^{\pm}(b_1,b_4,b_7,1)(J_{11}\cos k_1\tau+J_{12}\sin k_1\tau)$$

$$+f_2^{\pm}(b_2,b_5,b_8,1)+(J_{21}\cos k_2\tau+J_{22}\sin k_2\tau)$$

$$+f_2^{\pm}(b_3,b_6,b_9,1)(J_{31}\cos k_3\tau+J_{32}\sin k_3\tau)]\cos\beta\sin\lambda_m x$$

$$-\frac{8}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{n/2}}{m(n^2-1)}[f_2^{\pm}(b_1,b_4,b_7,n)(J_{11}\cos k_1\tau+J_{12}\sin k_1\tau)$$

$$+f_2^{\pm}(b_2,b_5,b_8,n)(J_{21}\cos k_2\tau+J_{22}\sin k_2\tau)$$

$$+f_2^{\pm}(b_3,b_6,b_9,n)(J_{31}\cos k_3\tau+J_{32}\sin k_3\tau)]\cos n\beta\sin\lambda_m x. \tag{25}$$

Considering the limiting conditions in which the impulse remains constant while its duration of action approaches zero, eqn (24) and eqn (25) become

$$\bar{\sigma}_1^{\pm} = \frac{4}{\pi^2}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{mQ}\left\{-v\lambda_m^2\left(\frac{\sin\alpha_2\tau}{\alpha_2}-\frac{\sin\alpha_1\tau}{\alpha_1}\right)\right.$$

$$\left.-\left(v\mp\frac{h}{2}\lambda_m^2\right)\left[(\lambda_m^2-\alpha_1^2)\frac{\sin\alpha_1\tau}{\alpha_1}-(\lambda_m^2-\alpha_2^2)\frac{\sin\alpha_2\tau}{\alpha_2}\right]\right\}\sin\lambda_m x$$

$$+\frac{2}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m}\left[f_1^{\pm}(b_1,b_4,b_7,1)\frac{\sin k_1\tau}{k_1}+f_1^{\pm}(b_2,b_5,b_8,1)\frac{\sin k_2\tau}{k_2}\right.$$

$$\left.+f_1^{\pm}(b_3,b_6,b_9,1)\frac{\sin k_3\tau}{k_3}\right]\cos\beta\sin\lambda_m x$$

$$-\frac{8}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{n/2}}{m(n^2-1)}\left[f_1^{\pm}(b_1,b_4,b_7,n)\frac{\sin k_1\tau}{k_1}\right.$$

$$\left.+f_1^{\pm}(b_2,b_5,b_8,n)\frac{\sin k_2\tau}{k_2}+f_1^{\pm}(b_3,b_6,b_9,n)\frac{\sin k_3\tau}{k_3}\right]\cos n\beta\sin\lambda_m x. \tag{26}$$

$$\bar{\sigma}_2^{\pm} = \frac{4}{\pi^2}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{mQ}\left\{-v^2\lambda_m^2\left(\frac{\sin\alpha_2\tau}{\alpha_2}-\frac{\sin\alpha_1\tau}{\alpha_1}\right)\right.$$

$$\left.-\left(1\mp\frac{h}{2}v\lambda_m^2\right)\left[(\lambda_m^2-\alpha_1^2)\frac{\sin\alpha_1\tau}{\alpha_1}-(\lambda_m^2-\alpha_2^2)\frac{\sin\alpha_2\tau}{\alpha_2}\right]\right\}\sin\lambda_m x$$

$$+\frac{2}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m}\left[f_2^{\pm}(b_1,b_4,b_7,1)\frac{\sin k_1\tau}{k_1}+f_2^{\pm}(b_2,b_5,b_8,1)\frac{\sin k_2\tau}{k_2}\right.$$

$$\left.+f_2^{\pm}(b_3,b_6,b_9,1)\frac{\sin k_3\tau}{k_3}\right]\cos\beta\sin\lambda_m x$$

$$-\frac{8}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{(-1)^{n/2}}{m(n^2-1)}\left[f_2^{\pm}(b_1,b_4,b_7,n)\frac{\sin k_1\tau}{k_1}\right.$$

$$\left.+f_1^{\pm}(b_2,b_5,b_8,n)\frac{\sin k_2\tau}{k_2}+f_2^{\pm}(b_3,b_6,b_9,n)\frac{\sin k_3\tau}{k_3}\right]\cos n\beta\sin\lambda_m x. \tag{27}$$

3.2. *Example B*

In this example, the impulsive loading is suddenly applied inward over a small area $x_1 \leqslant x \leqslant x_2$ in the axial direction, and $-\beta_0 \leqslant \beta \leqslant \beta_0$ in the circumferential direction of the outer surface of the shell. The load is assumed to be constant within the time duration $0 \leqslant t \leqslant t_1$ and vanish when $t > t_1$ (Fig. 3), which can be expressed as

$$p'(x,\beta,t) = \begin{cases} p'_0, & 0\leqslant t\leqslant t_1,\ x_1\leqslant x\leqslant x_2,\ -\beta_0\leqslant\beta\leqslant\beta_0; \\ 0, & \text{otherwise.}\end{cases} \tag{28}$$

Using the same convention as in Example A, the dimensionless transformed form of loading is

$$p(x,\beta,\tau) = \begin{cases} p_0, & x_1\leqslant x\leqslant x_2,\ -\beta_0\leqslant\beta\leqslant\beta_0,\ 0\leqslant\tau\leqslant\tau_1; \\ 0, & \text{otherwise.}\end{cases} \tag{29}$$

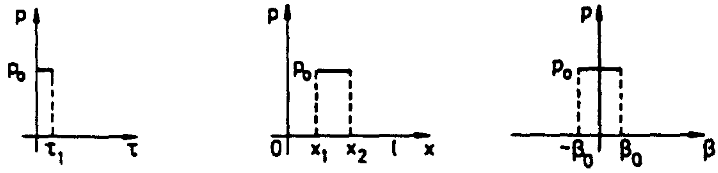

Fig. 3. Impact loading as a function of time and position (Example B).

where

$$p_0 = \frac{I}{\tau_1} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_n c_m \cos n\beta \sin \lambda_m x.$$

The coefficients for the double Fourier series expansion for $p(x, \beta, t)$ are given by

$$a_0 = \frac{\beta_0}{\pi}$$

$$a_n = \frac{2}{n\pi} \sin n\beta_0 \qquad (n = 1, 2, 3, \ldots),$$

$$c_m = \frac{4}{m\pi} \sin \frac{x_1 + x_2}{2l} m\pi \sin \frac{x_2 - x_1}{2l} m\pi \qquad (m = 1, 2, 3, \ldots).$$

Then, following the same procedures described in Example A, the stress responses when $0 \leqslant \tau \leqslant \tau_1$ are

$$\bar{\sigma}_1^{\pm} = \frac{4\beta_0}{\pi^2 \tau_1} \sum_{m=1,2,3,\ldots}^{\infty} \frac{1}{mQ} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi \left\{ -\nu\lambda_m^2 \left(\frac{1-\cos\alpha_2\tau}{\alpha_2^2} - \frac{1-\cos\alpha_1\tau}{\alpha_1^2}\right) \right.$$
$$\left. -\left(\nu \mp \frac{h}{2}\lambda_m^2\right)\left[(\lambda_m^2-\alpha_1^2)\frac{1-\cos\alpha_1\tau}{\alpha_1^2} - (\lambda_m^2-\alpha_2^2)\frac{1-\cos\alpha_2\tau}{\alpha_2^2}\right]\right\} \sin\lambda_m x$$
$$+ \frac{8}{\pi^2\tau_1} \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \frac{\sin n\beta_0}{mn} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi$$
$$\times \left[f_1^{\pm}(b_1, b_4, b_7, n) \frac{1-\cos k_1\tau}{k_1^2} + f_1^{\pm}(b_2, b_5, b_8, n) \frac{1-\cos k_2\tau}{k_2^2} \right.$$
$$\left. + f_1^{\pm}(b_3, b_6, b_9, n) \frac{1-\cos k_3\tau}{k_3^2} \right] \cos n\beta \sin \lambda_m x. \tag{30}$$

$$\bar{\sigma}_2^{\pm} = \frac{4\beta_0}{\pi^2 \tau_1} \sum_{m=1,2,3,\ldots}^{\infty} \frac{1}{mQ} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi \left\{ -\nu^2\lambda_m^2 \left(\frac{1-\cos\alpha_2\tau}{\alpha_2^2} - \frac{1-\cos\alpha_1\tau}{\alpha_1^2}\right) \right.$$
$$\left. -\left(1 \mp \frac{h}{2}\nu\lambda_m^2\right)\left[(\lambda_m^2-\alpha_1^2)\frac{1-\cos\alpha_1\tau}{\alpha_1^2} - (\lambda_m^2-\alpha_2^2)\frac{1-\cos\alpha_2\tau}{\alpha_2^2}\right]\right\} \sin\lambda_m x$$
$$+ \frac{8}{\pi^2\tau_1} \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \frac{\sin n\beta_0}{mn} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi$$
$$\times \left[f_2^{\pm}(b_1, b_4, b_7, n) \frac{1-\cos k_1\tau}{k_1^2} + f_2^{\pm}(b_2, b_5, b_8, n) \frac{1-\cos k_2\tau}{k_2^2} \right.$$
$$\left. + f_2^{\pm}(b_3, b_6, b_9, n) \frac{1-\cos k_3\tau}{k_3^2} \right] \cos n\beta \sin \lambda_m x. \tag{31}$$

For $\tau \geq \tau_1$, the stress responses are

$$\bar{\sigma}_1^{\pm} = \frac{4\beta_0}{\pi^2} \sum_{m=1,2,3,\ldots}^{\infty} \frac{1}{mQ} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi \left\{ -v\lambda_m^2[(S_{21}\cos\alpha_2\tau + S_{22}\sin\alpha_2\tau) - (S_{11}\cos\alpha_1\tau + S_{12}\sin\alpha_1\tau)] - \left(v \mp \frac{h}{2}\lambda_m^2\right)[(\lambda_m^2-\alpha_1^2)(S_{11}\cos\alpha_1\tau + S_{12}\sin\alpha_1\tau) - (\lambda_m^2-\alpha_2^2)(S_{21}\cos\alpha_2\tau + S_{22}\sin\alpha_2\tau)]\right\} \sin\lambda_m x$$
$$+ \frac{8}{\pi^2} \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \frac{\sin n\beta_0}{mn} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi \times [f_1^{\pm}(b_1,b_4,b_7,n)(J_{11}\cos k_1\tau + J_{12}\sin k_1\tau) + f_1^{\pm}(b_2,b_5,b_8,n)(J_{21}\cos k_2\tau + J_{22}\sin k_2\tau) + f_1^{\pm}(b_3,b_6,b_9,n)(J_{31}\cos k_3\tau + J_{32}\sin k_3\tau)] \cos n\beta \sin\lambda_m x. \quad (32)$$

$$\bar{\sigma}_2^{\pm} = \frac{4\beta_0}{\pi^2} \sum_{m=1,2,3,\ldots}^{\infty} \frac{1}{mQ} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi \left\{ -v^2\lambda_m^2[(S_{21}\cos\alpha_2\tau + S_{22}\sin\alpha_2\tau) - (S_{11}\cos\alpha_1\tau + S_{12}\sin\alpha_1\tau)] - \left(1 \mp \frac{h}{2}v\lambda_m^2\right)[(\lambda_m^2-\alpha_1^2)(S_{11}\cos\alpha_1\tau + S_{12}\sin\alpha_1\tau) - (\lambda_m^2-\alpha_2^2)(S_{21}\cos\alpha_2\tau + S_{22}\sin\alpha_2\tau)]\right\} \sin\lambda_m x$$
$$+ \frac{8}{\pi^2} \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \frac{\sin n\beta_0}{mn} \sin\left(\frac{x_1+x_2}{2l}\right) m\pi \sin\left(\frac{x_2-x_1}{2l}\right) m\pi \times [f_2^{\pm}(b_1,b_4,b_7,n)(J_{11}\cos k_1\tau + J_{12}\sin k_1\tau) + f_2^{\pm}(b_2,b_5,b_8,n)(J_{21}\cos k_2\tau + J_{22}\sin k_2\tau) + f_2^{\pm}(b_3,b_6,b_9,n)(J_{31}\cos k_3\tau + J_{32}\sin k_3\tau)] \cos n\beta \sin\lambda_m x, \quad (33)$$

where S_{ij}, $(i, j = 1, 2)$ and J_{ij}, $(i = 1, 2, j = 1, 2, 3)$ are same as for eqn (22).

4. DISCUSSION

By systematically changing parameters, the axial stress $\sigma_1^{\pm}$ and the circumferential stress $\sigma_2^{\pm}$ for all given values can be computed. The terms of the double series have been summed to $n = 100$, $m = 101$ for time $\tau \leq 4\pi$ at position $x = 0.5$, $\beta = 0$ for the parameter values $l = 2, 5, 10, 20$ and $h = 0.01, 0.05$ and impact duration $\tau_1 = 0$. Figures 4–5 show the results of the calculations.

The effect of the length is illustrated in Figs 4a–4d for the case of a thin-walled cylinder of $h = 0.01$, subjected to a pure impulse of very short duration ($\tau_1 = 0$). The stresses have been calculated at the mid-span ($x = 0.5$, $\beta = 0$). For a long shell, with $l = 20$, there is good agreement between the present results and those of Humphreys and Winter (1965), obtained assuming that there was no variation in the state of stress with axial position. This condition corresponds to $\sigma_1 = 0$. As the cylinder becomes shorter, the overall stress–time response changes and small wavelength oscillations are superimposed on to the lower frequency variation. The effect is particularly pronounced for the axial stresses and is

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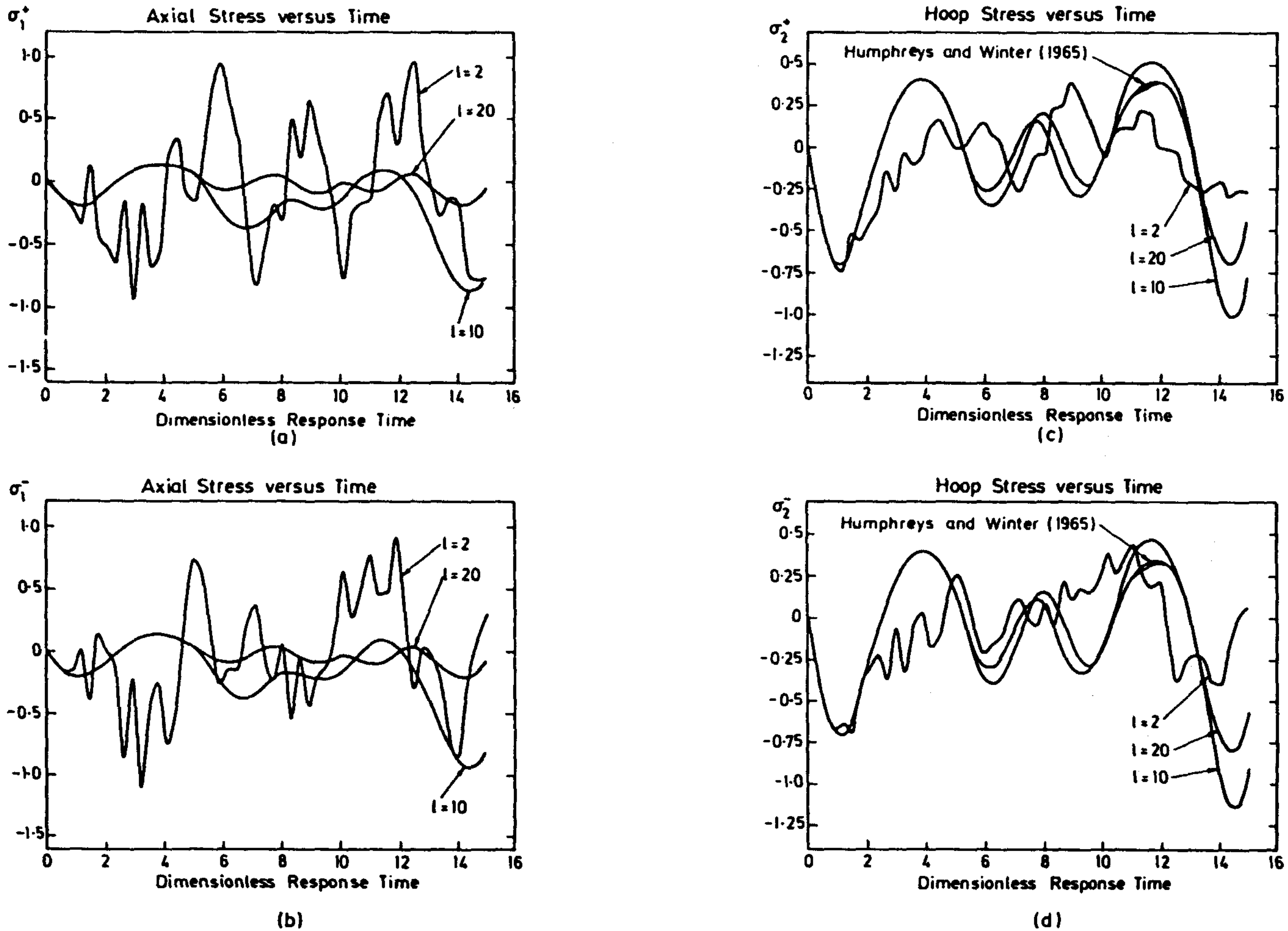

Fig. 4. Impact response (Example A): stresses at mid-span ($x = 0.5$, $\beta = 0$) for a cylinder of $h = 0.01$, $\tau_1 = 0$.

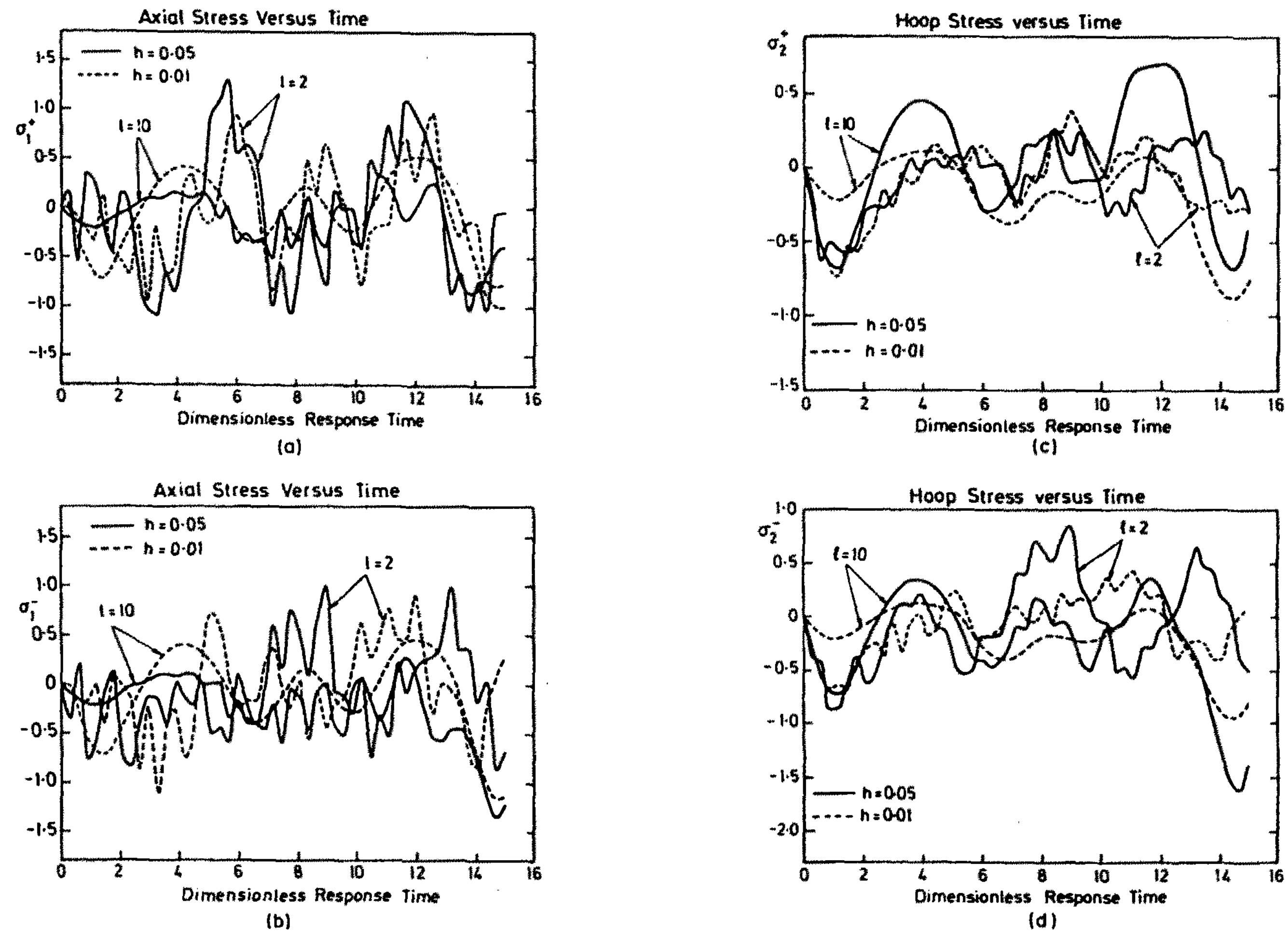

Fig. 5. Impact response (Example A): stresses at mid-span ($x = 0.5$, $\beta = 0$) for a cylinder of $\tau_1 = 0$.

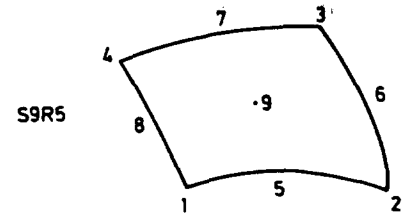

Fig. 6. S9R5 shell element of ABAQUS.

ascribed to the stress wave reflections at the ends. Ignoring the end effects, a short shell ($l = 2$) would grossly underestimate the values of the stress, since axial stresses ranging between -1 and $+1$ become predominant over hoop stresses ranging from -0.75 to $+0.5$ (Humphreys and Winter, 1965).

The effect of thickness is illustrated in Fig. 5. There are obvious bending effects in the stress–time history, the thicker the shell, the higher the amplitudes of stresses.

Example B has been calculated for a cylindrical shell with a thickness ratio of 0.01 and length-to-radius ratio of 5. The load was applied over a rectangular pad defined by $x_1 = 0.4$, $x_2 = 0.6$ and $\beta_0 = 15°$ (see Fig. 3). For comparison, the problem was also solved using ABAQUS, employing 240[10×24] S9R5 shell elements, as plotted in Fig. 6. The deformed shape is shown in Fig. 7. Figures 8 and 9 show the variation of dimensionless stresses with position. The terms of the double trigonometric series have been summed to $n = 100$ and $m = 100$. There is excellent agreement between ABAQUS and the present solution.

The double trigonometric series finds its merit to treat boundary condition problems, despite that it is not monotonically convergent. The convergence of the double trigonometric series employed was studied by taking up to 5000×5000 terms. It was found that each result oscillated about a mean value and that the amplitude of oscillation decreased with the number of terms in the series but depended on the position of stress that was being calculated, and on the type of loading. In addition, the axial stress always oscillates more seriously than the circumferential stress because of the traction on the two ends. Typically, in the second problem at the center of loading where $x = 0.5$ and $\beta = 0$, taking between 30×30 and 60×60 terms gave an amplitude of oscillation of $\pm 3.5\%$ above the average for the axial stress and $\pm 2.1\%$ for the circumferential stress; taking between 100×100 and 120×120 terms reduced the amplitude to $\pm 0.25\%$ for the axial stress and $\pm 0.15\%$ for the circumferential stress. In either case, the difference between the average results was less than 2%. The solution of the second example was obtained with 100×100 terms which requires 70 min CPU on a Micro-VAX II computer, against more than 5 hours CPU for the ABAQUS run. If only the response of a fixed position at a fixed time is needed, the CPU involved is only of the order of tens of seconds.

5. CONCLUSIONS

The exact solution of the elastic transient response of a cylindrical shell subject to a transverse impact loading has been obtained in closed form. The treatment is valid for a

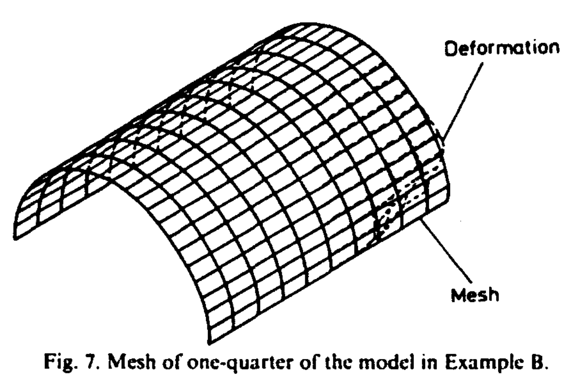

Fig. 7. Mesh of one-quarter of the model in Example B.

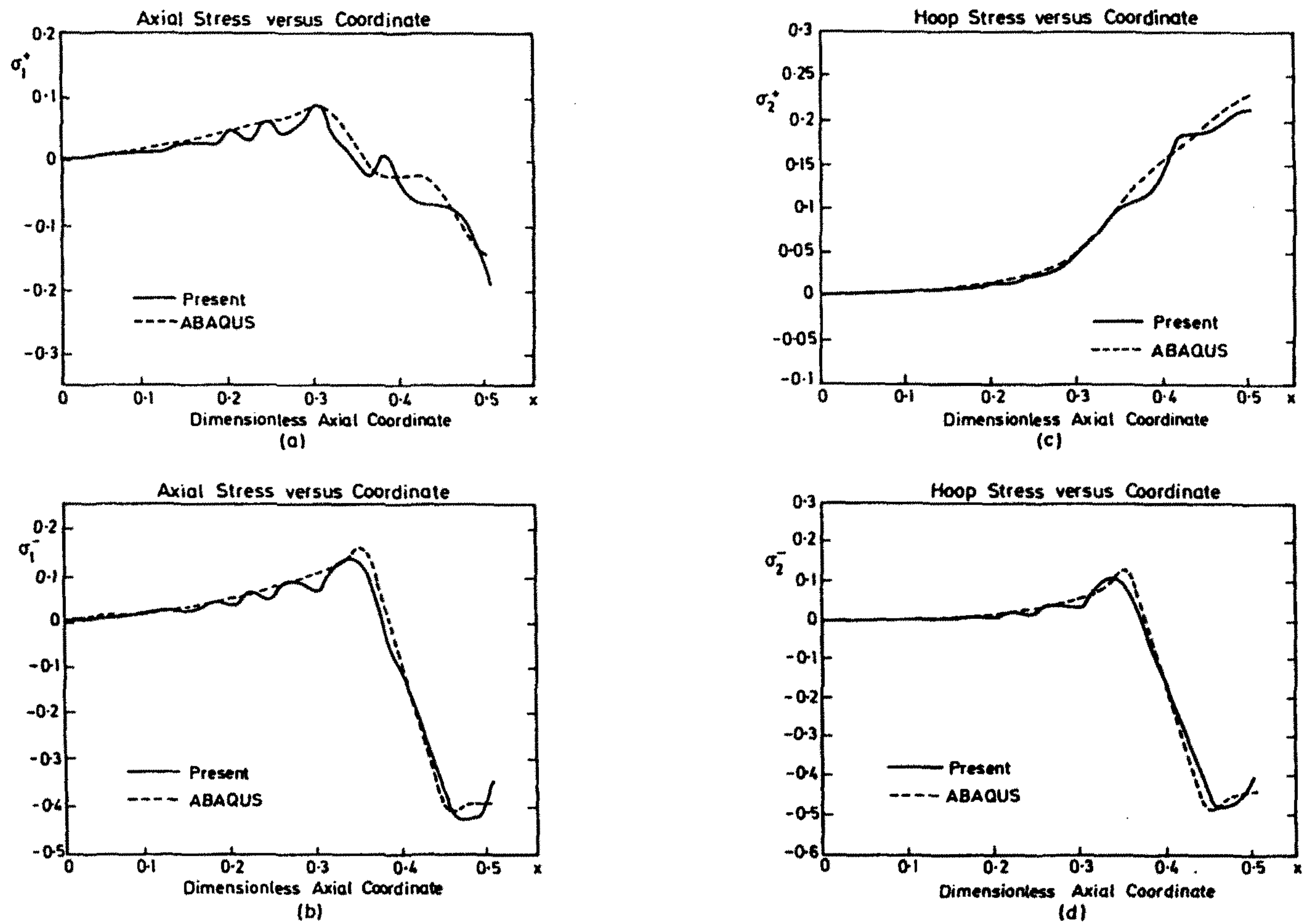

Fig. 8. Comparison of stresses in axial direction ($\beta = 0$) between present analysis (Example B) and ABAQUS [$\tau_1 = 3.981147$ (30 μs), $\tau = 5.308197$ (40 μs), $h = 0.01$, $l = 5$].

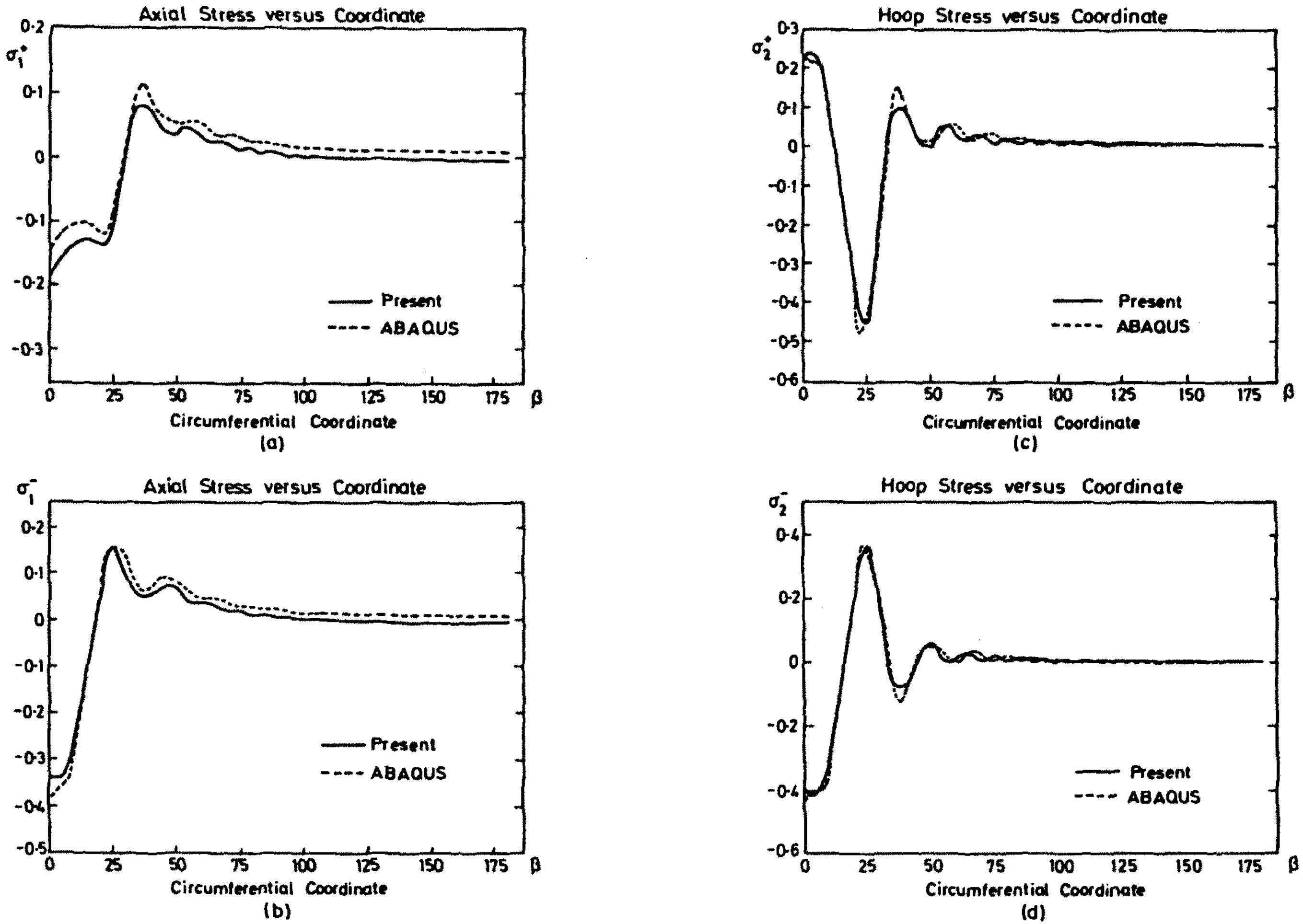

Fig. 9. Comparison of stresses in circumferential direction ($x = 0$) between present analysis (Example B) and ABAQUS [$\tau_1 = 3.981147$ (30 μs), $\tau = 5.308197$ (40 μs), $h = 0.01$, $l = 5$].

cylinder of any length, with simply supported ends and is therefore an improvement over other published theoretical solutions, which are only valid for infinitely long cylinders. The effect of the proximity of the ends is particularly important in building up high axial stress that may result in failure. In both the examples that have been treated, very high bending stresses have been obtained. The results have been shown to be in excellent agreement with ABAQUS.

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APPENDIX

We make use of the Laplace transform technique to solve the simultaneous differential eqns (10) by defining

$$\bar{U}(s) = \int_0^\infty e^{-s\tau} U(n,m,\tau)\,d\tau$$

$$\bar{V}(s) = \int_0^\infty e^{-s\tau} V(n,m,\tau)\,d\tau$$

$$\bar{W}(s) = \int_0^\infty e^{-s\tau} W(n,m,\tau)\,d\tau$$

$$\bar{P}(s) = \int_0^\infty e^{-s\tau} P(n,m,\tau)\,d\tau. \tag{34}$$

Then the differential equation can be transformed into a linear algebraic system of simultaneous equations, when $n \geq 1$

$$(\mathbf{A} - s^2\mathbf{E})\bar{\mathbf{W}} = -\bar{\mathbf{P}}(s) \tag{35}$$

where $\mathbf{E}$ is an order three unit matrix. The algebraic solutions of eqn (35) are

$$\bar{U} = \Delta_U/\Delta = \left(\frac{b_1}{s^2+k_1^2} + \frac{b_2}{s^2+k_2^2} + \frac{b_3}{s^2+k_3^2}\right)\bar{P}(s)$$

$$\bar{V} = \Delta_V/\Delta = \left(\frac{b_4}{s^2+k_1^2} + \frac{b_5}{s^2+k_2^2} + \frac{b_6}{s^2+k_3^2}\right)\bar{P}(s)$$

$$\bar{W} = \Delta_W/\Delta = \left(\frac{b_7}{s^2+k_1^2} + \frac{b_8}{s^2+k_2^2} + \frac{b_9}{s^2+k_3^2}\right)\bar{P}(s), \tag{36}$$

where

$$\Delta = |\mathbf{A} - s^2\mathbf{E}| = (s^2+k_1^2)(s^2+k_2^2)(s^2+k_3^2). \tag{37}$$

We can see that it will be convenient to obtain the inverse Laplace transform by expressing Δ_U/Δ, Δ_V/Δ and Δ_W/Δ in the form containing $b_1/(s^2+k_1^2)$, $b_2/(s^2+k_2^2)$ and $b_3/(s^2+k_3^2)$, etc. If we denote | | as the relevant value of the determinant, then a series of coefficients appearing in eqn (36) are listed below

$$\Delta_U = \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22}-s^2 & a_{23} \\ -\bar{P}(s) & a_{32} & a_{33}-s^2 \end{vmatrix} \quad \Delta_V = \begin{vmatrix} a_{11}-s^2 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & -\bar{P}(s) & a_{33}-s^2 \end{vmatrix} \quad \Delta_W = \begin{vmatrix} a_{11}-s^2 & a_{12} & 0 \\ a_{21} & a_{22}-s^2 & 0 \\ a_{31} & a_{32} & -\bar{P}(s) \end{vmatrix}$$

and k_i^2 ($i = 1, 2, 3$) are obtained by cubic root formulae [from eqn (37)]

$$k_1^2 = \frac{g_1}{3} + 2\sqrt[3]{r}\cos\theta$$

$$k_2^2 = \frac{g_1}{3} + 2\sqrt[3]{r}\cos\left(\theta+\frac{2}{3}\pi\right)$$

$$k_3^2 = \frac{g_1}{3} + 2\sqrt[3]{r}\cos\left(\theta+\frac{4}{3}\pi\right).$$

where

$$r = \sqrt{-(l_2/3)^3}\,;\ \theta = \frac{1}{3}\cos^{-1}\left(-\frac{l_1}{2r}\right)$$

$$l_1 = \frac{1}{3}g_1g_2 - \frac{2}{27}g_1^3 - g_3\,;\ l_2 = g_2 - \frac{1}{3}g_1^2$$

$$g_1 = -(a_{11}+a_{22}+a_{33})$$

$$g_2 = a_{11}a_{22}+a_{22}a_{33}+a_{33}a_{11}-(a_{12}a_{21}+a_{23}a_{32}+a_{13}a_{31})$$

$$g_3 = a_{11}a_{23}a_{32}+a_{13}a_{22}a_{31}+a_{12}a_{21}a_{33}-(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}).$$

The coefficients b_i ($i = 1, 2, \ldots, 9$) are obtained by finding the solution of linear algebraic equations of the manipulations of the polynomials in eqn (36), hence we have

$$b_i = \Delta_i/\Delta_0 \quad (i = 1, 2, \ldots, 9).$$

where

$$\Delta_0 = \begin{vmatrix} 1 & 1 & 1 \\ k_2^2+k_3^2 & k_1^2+k_3^2 & k_1^2+k_2^2 \\ k_2^2k_3^2 & k_1^2k_3^2 & k_1^2k_2^2 \end{vmatrix} \quad \Delta_1 = \begin{vmatrix} 0 & 1 & 1 \\ a_{11} & k_1^2+k_3^2 & k_1^2+k_2^2 \\ a_{12}a_{21}-a_{11}a_{22} & k_1^2k_3^2 & k_1^2k_2^2 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ k_2^2+k_3^2 & a_{11} & k_1^2+k_2^2 \\ k_2^2k_3^2 & a_{12}a_{21}-a_{11}a_{22} & k_1^2k_2^2 \end{vmatrix} \quad \Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ k_2^2+k_3^2 & k_1^2+k_3^2 & a_{11} \\ k_2^2k_3^2 & k_1^2k_3^2 & a_{12}a_{21}-a_{11}a_{22} \end{vmatrix}$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 \\ a_{21} & k_1^2+k_3^2 & k_1^2+k_2^2 \\ a_{13}a_{21}-a_{11}a_{23} & k_1^2k_3^2 & k_1^2k_2^2 \end{vmatrix} \quad \Delta_5 = \begin{vmatrix} 1 & 0 & 1 \\ k_2^2k_3^2 & a_{23} & k_1^2+k_2^2 \\ k_2^2k_3^2 & a_{13}a_{21}-a_{11}a_{23} & k_1^2k_2^2 \end{vmatrix}$$

$$\Delta_6 = \begin{vmatrix} 1 & 1 & 0 \\ k_2^2+k_3^2 & k_1^2+k_3^2 & a_{23} \\ k_2^2k_3^2 & k_1^2k_3^2 & a_{13}a_{21}-a_{11}a_{23} \end{vmatrix} \quad \Delta_7 = \begin{vmatrix} 1 & 1 & 1 \\ -(a_{11}+a_{22}) & k_1^2+k_3^2 & k_1^2+k_2^2 \\ a_{11}a_{22}-a_{12}a_{21} & k_1^2k_3^2 & k_1^2k_2^2 \end{vmatrix}$$

$$\Delta_8 = \begin{vmatrix} 1 & 1 & 1 \\ k_2^2+k_3^2 & -(a_{11}+a_{22}) & k_1^2+k_2^2 \\ k_2^2k_3^2 & a_{11}a_{22}-a_{12}a_{21} & k_1^2k_2^2 \end{vmatrix} \quad \Delta_9 = \begin{vmatrix} 1 & 1 & 1 \\ k_2^2+k_3^2 & k_1^2+k_3^2 & -(a_{11}+a_{22}) \\ k_2^2k_3^2 & k_1^2k_3^2 & a_{11}a_{22}-a_{12}a_{21} \end{vmatrix}$$

So, inverting eqn (36), the general solution can be expressed as eqn (12).